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Morita equivalence and T-duality (or B versus Θ)

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ABSTRACT: T-duality in M(atrix) theory has been argued to be realized as Morita equivalence in Yang-Mills theory on a non-commutative torus (NCSYM). Even though the two have the same structure group, they differ in their action since Morita equivalence makes crucial use of an additional modulus on the NCSYM side, the constant abelian magnetic background. In this paper, we reanalyze and clarify the correspondence between M(atrix) theory and NCSYM, and provide two resolutions of this puzzle. In the first of them, the standard map is kept and the extra modulus is ignored, but the anomalous transformation is offset by the M(atrix) theory "rest term". In the second, the standard map is modified so that the duality transformations agree, and a SO(d) symmetry is found to eliminate the spurious modulus. We argue that this is a true symmetry of supersymmetric Born-Infeld theory on a non-commutative torus, which allows to freely trade a constant magnetic background for non-commutativity of the base-space. We also obtain a BPS mass formula for this theory, invariant under T-duality, U-duality, and continuous SO(d) symmetry.

KEYWORDS: M-Theory, D-branes, String Duality, M(atrix) Theories.

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Contents

Gauge theories on non-commutative spaces arise in M(atrix) theory compactifications with non-vanishing three-form $\mathcal{C}_{ij-}$ on the light-cone [1], and in string theory as world-volume theories of D-branes in large B-field [2]. This effect is most easily understood in toroidal compactifications after T-duality: the torus with large B-field turns into a slanted torus, and the non-locality characteristic of non-commutative geometry arises from the shortest open strings relating the D-brane to its translated images as the torus becomes more slanted [2, 3]. In fact, in the presence of a non-vanishing B-field, — or U(1) field strength, since only the combination $F + B$ is gauge-invariant — the longitudinal coordinates at the end of the open strings attached to the D-brane become non-commutative [4, 5, 6], revealing that the world-volume itself has turned into a non-commutative space. Therefore, the worldvolume theory of D-branes should described by Born-Infeld theory on a non-commutative space [7]. Unfortunately, this theory lacks a precise definition, for the same reason as non-abelian Born-Infeld theory does. In this work, we shall follow a rather indirect route to uncover some of its properties.

The dynamics of N D0-branes on T^d in the Sen-Seiberg limit of vanishing volume [8, 9] is supposed to describe M-theory on the light-cone [10, 11] and the background B-field, hold fixed in the limit, simply corresponds to the vev of the three-form field strength C_{ij-} on the lightlike direction [1]. The BPS mass formula on the M-theory side is known from U-duality considerations [22] (see [12, 13] for a review), whereas the BPS energy spectrum in the non-commutative super-Yang-Mills theory (NCSYM) was derived rigorously in [14]. This theory is invariant under a $\mathrm{SO}(d, d, \mathbb{Z})$ group generated by Morita equivalence, which was argued to reflect the T-duality invariance of M-theory on a torus T^{d+1} [1, 15]. Further studies have followed in [16, 17, 18, 19]. The BPS spectrum even organizes itself in multiplets of the extended U-duality group $E_{d+1(d+1)}(\mathbb{Z})$, although only T-duality is a symmetry of the mass formula [20, 21, 22, 23]. The correspondence between the M-theory and gauge theory sides nevertheless encounters several difficulties. For one thing, the NC-SYM theory seems to offer an extra modulus Φ_{ij}, which corresponds to a constant abelian magnetic background, and for another, T-duality and Morita equivalence

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act differently on the two sides, even though representing the action of the same $\mathrm{SO}(d,d,\mathbb{Z})$ group. Our first aim in this paper is to reconcile the two sides and clarify this correspondence.

Our first approach takes advantage of the fact that the abelian magnetic background couples to the Yang-Mills fields through topological terms, and thus can be set to zero upon proper account of the induced spectral flow. The resulting theory is no longer invariant under Morita equivalence, but we shall argue that the anomalous variation is offset by a "rest term" which is naturally motivated from the M(atrix) side. In this way, we recover the standard mapping between M-theory and NCSYM, and confirm the identification of the M-theory B-field with the NCSYM deformation parameter Θ. We also obtain a simpler relation between NCSYM energies and M-theory masses than usually presented. This will be discussed in section 1.

In our second approach, we pay full respect to the dual parameters Θ and Φ of the NCSYM, and exhibit a map to the M-theory B-field modulus which translates Morita equivalence into the standard T-duality. This map is not one-to-one, and indeed we uncover a continuous $\mathrm{SO}(d)$ symmetry relating the various preimages of a given B. These map and symmetry can be found in equations (2.10), (2.11) in section 2 of the present paper for the $d=2$ case, and (2.24), (2.25) for the general case. We propose that this invariance is a true symmetry of the putative Born-Infeld theory on a non-commutative torus, and exhibit an T-duality and U-duality invariant mass formula (2.26) which reduces to the NCSYM mass formula in the "non-relativistic" $l_s \to 0$ limit (in the sense of [24]), and to the usual Born-Infeld mass formula for a commutative torus. Our result implies that a magnetic background Φ can be freely traded for a non-commutativity Θ of the base-space, through a $\mathrm{SO}(d)$ transformation. This equivalence between base-space non-commutativity and magnetic background fits nicely with the observations reported in [25]. Unfortunately, for lack of an appropriate formulation of non-commutative Born-Infeld theory, proving that this invariance is a full-fledged symmetry will remain beyond reach.

1. M-theory and M(atrix) gauge theory

M-theory mass spectrum and T-duality. The BPS mass formula for M-theory compactified on a torus T^d times a circle of radius R_s in the presence of general gauge backgrounds was obtained in [22] from U-duality requirements. For simplicity, we shall at first focus on the $d=2$ case, where it shrinks to a more manageable form,

$$\begin{aligned}\mathcal{M}^2 =& \frac{1}{R_s^2}(m_s + B_{ij}m^{ij})^2 + \left[(m_i + B_{ij}m^{js}) + A_i(m_s + B_{jk}m^{jk})\right]^2 + \\ &+ \frac{(m^{ij})^2}{l_M^6} + \frac{R_s^2}{l_M^6}\left(m^{is} + A_j m^{ij}\right)^2 + \sqrt{\frac{(K^i)^2 + R_s^2(K^s + A_i K^i)^2}{l_M^6}}\end{aligned} \tag{1.1}$$

Here, l_M denotes the 11D Planck length, A_i and B_{ij} the off-diagonal metric G_{is} and three-form C_{ijs} where s stands for the direction of radius R_s, and the contractions are performed with respect to the metric G_{ij} or inverse metric G^{ij} on the torus T^d. The integers m_s, m_i, m^{is}, m^{ij} and K^i, K^s stem from the reduction of the charges m_I, m^{IJ} and K^I of the particle and string multiplets on $T^2 \times S^1$, respectively. In particular, the string charges are related to the particle charges through $K^I = m_J m^{IJ}$ for 1/4-BPS states, which has to vanish for 1/2-BPS states. Their physical interpretation is clear from their contributions to the mass: m_s and m_i are the momenta around S_1 and T^2, m^{is} and m^{ij} are the membrane charge on $S^1 \times T^2$. This mass formula is invariant under the U-duality group $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$ on $T^2 \times S^1 = T^3$, under which the momentum multiplet charges transforms as $(3,2)$ and the string charges as $(3,1)$. We shall be particularly interested in the T-duality subgroup $\mathrm{SL}(2,\mathbb{Z})_T \subset \mathrm{SL}(3,\mathbb{Z})$, whose action is most easily seen after going to string theory variables $l_s^2 = l_M^3/R_s$, $g_s l_s = R_s$:

$$\begin{aligned}\mathcal{M}^2 =& \frac{T_2}{g_s^2 l_s^2} \frac{|m-nT|^2}{T_2} + \sqrt{\frac{T_2}{g_s^2 l_s^4} K^i \hat{G}_{ij} K^j + \frac{K^2}{l_s^4}} + \\ &+ \frac{1}{l_s^2 T_2} \left[(m_i + A_i m) - (n_i + A_i n) T\right] \hat{G}^{ij} \left[(m_j + A_j m) - (n_j + A_j n)\bar{T}\right] \end{aligned} \tag{1.2}$$

where $T = T_1 + iT_2 = B_{12} + i\sqrt{\det G_{ij}}/l_s^2$ is the standard complex modulus and $\hat{G}_{ij} = G_{ij}/\sqrt{\det G}$ is the unit volume metric of the torus T^2. We also relabelled the charges as $m = m_s, n = \epsilon_{ij} m^{ij}/2, n_i = \epsilon_{ij} m^{js}, K = K^s = \epsilon^{ij} m_i n_j, K^i = \epsilon^{ij}(m n_i - n m_i)$. The mass formula is then invariant under the standard $\mathrm{SL}(2,\mathbb{Z})_T$ T-duality

$$T \to \frac{aT+b}{cT+d}, \qquad g_s \to \frac{g_s}{|cT+d|}, \qquad \begin{pmatrix} m & m_i \\ n & n_i \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m & m_i \\ n & n_i \end{pmatrix}, \tag{1.3}$$

where $ad - bc = 1$, leaving $\hat{G}_{ij}$ and k_i invariant. Note for later reference the following useful consequences:

$$T_1 \to \frac{aT_1+b}{cT_1+d} + \frac{cT_2^2}{(cT_1+d)|cT+d|^2}, \qquad T_2 \to \frac{T_2}{|cT+d|^2}, \qquad m - nT \to \frac{m-nT}{cT_1+d} \tag{1.4}$$

Following Sen and Seiberg [8, 9], the Discrete Light-Cone Quantization of M-theory on a torus T^d with light-cone momentum m is equivalent to M-theory compactified on T^d times a space-like circle of radius R_s with momentum m, in the scaling limit $R_s \to 0$,

$$l_s = \frac{R_s^{1/2}}{M^{3/2}}, \qquad g_s = (R_s M)^{3/4}, \quad G_{ij} = \frac{R_s}{M} \gamma_{ij}, \tag{1.5}$$

holding M and γ_{ij} fixed. Under this scaling, the mass formula reduces to

$$\begin{aligned}\mathcal{M} =& \frac{|m-nB|}{g_s l_s} + \frac{T_2^2 n^2}{2g_s l_s|m-nB|} + \\ &+ \frac{g_s}{2l_s T_2|m-nB|}\Big[(m_i+A_i m)-(n_i+A_i n)B\Big]\hat{G}^{ij}\Big[(m_j+A_j m)-(n_j+A_j n)B\Big] + \\ &+ \frac{\sqrt{T_2}}{l_s|m-nB|}\sqrt{K^i\hat{G}_{ij}K^j} + \mathcal{O}(R_s)\end{aligned} \tag{1.6}$$

while the interactions between the D0-particles decouple from the closed string bulk modes, and when $A = B = 0$ reduce to a U(m) super-Yang-Mills theory on the dual torus. In this limit, the relativistic dispersion relation (1.2) has turned into its non-relativistic Galilean limit (1.6).

Non-commutative Yang-Mills and Morita equivalence. On the other hand, the BPS energy formula of super-Yang-Mills theory on a non-commutative torus T_θ^2 given by the action functional[1]

$$S = -\frac{V}{4g_{YM}^2}\mathrm{Tr}\ (F_{ij}+\Phi_{ij}\cdot 1)^2 + i\frac{V}{2g_{YM}^2}\mathrm{Tr}\ \bar{\psi}\Gamma^i[\nabla_i,\psi] + \lambda^i \mathrm{Tr}\ F_{0i} \tag{1.7}$$

where $\Phi_{ij} = \phi\epsilon_{ij}$, was computed in [14] from a study of the central charges in the BPS algebra, with the result that

$$\begin{aligned}E =& \frac{1}{2Vg_{YM}^2\dim E}(q+\phi\dim E)^2 + \\ &+ \frac{g_{YM}^2}{2V\dim E}(n^i-\theta m^i+\lambda^i\dim E)^2 + \frac{1}{\dim E}\sqrt{(k_i)^2}\,,\end{aligned} \tag{1.8}$$

with $\dim E = p - q\theta$ and $k_i = \epsilon_{ij}(pm^i - qn^i)$. Contractions are performed with the metric g_{ij} or inverse metric g^{ij} depending on the position of the indices. This formula is invariant under the $\mathrm{SL}(2,Z)_M$ action

$$\begin{aligned}&\theta\to\frac{a\theta+b}{c\theta+d}\,, \qquad \phi\to(c\theta+d)^2\phi - c(c\theta+d)\,, \qquad g_{ij}\to(c\theta+d)^2 g_{ij}\,, \\ &g_{YM}^2\to(c\theta+d)g_{YM}^2\,, \qquad \begin{pmatrix}p & n^i\\ q & m^i\end{pmatrix}\to\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}p & n^i\\ q & m^i\end{pmatrix}\,,\end{aligned} \tag{1.9}$$

leaving k_i invariant. It is useful to note that this transformation implies

$$V\to(c\theta+d)^2 V\,, \qquad \dim E\to\frac{\dim E}{c\theta+d}\,, \qquad \mathrm{mid}\,E\to\mathrm{mid}\,E(c\theta+d) \tag{1.10}$$

where we suggestively defined $\mathrm{mid}\,E = q + \phi\dim E$. This invariance was deduced in [15] from Morita equivalence of non-commutative tori T_θ. It is important to notice that this transformation involves the magnetic background field ϕ, as also recognized in [1, 16].

[1] The trace Tr for a commutative torus reduces to the integral $\int d^dx/V$.

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Morita equivalence and T-duality reconciled. We now would like to reconcile the M-theory mass formula (1.6) and NCSYM mass formula (1.8). The two certainly look very similar, but there are some important differences. First of all, the first term in (1.6), linear in the charges, has no counterpart in (1.8): it may be included on the gauge theory side by adding an innocuous term

$$S_{rest} = \frac{V}{g_{YM}^2 l_s^4} \mathrm{Tr}\ 1\ , \tag{1.11}$$

to the action functional (1.7). Indeed, this term is simply the constant term in the Born-Infeld action, and gives the mass of the m Dd-branes on which the gauge theory lives. Such a linear term was also introduced by hand in [1] to enforce invariance. We will find more hints for the relevance of Born-Infeld theory shortly.

The second major difference is in the extra modulus ϕ which appears on NCSYM side and has no obvious counterpart on the M-theory side. When $\phi = 0$ and up to the rest term, we obtain agreement under the map

$$\begin{gathered}(p,q) = (m,n)\,, \qquad (n^i, m^i) = (m_i, n_i)\,, \qquad k_i = K^i\,, \\ g_{ij} = l_s^4 G^{ij}\,, \qquad g_{YM}^2 = \frac{g_s l_s^{2d-3}}{\sqrt{\det(G_{ij})}}\,, \qquad \theta = B\,, \qquad \lambda^i = A_i\end{gathered} \tag{1.12}$$

which implies $\hat{g}_{ij} = \hat{G}^{ij}$, $V = l_s^2/T_2$ and reduces to the standard map at $B = \theta = 0$. However, we should still understand how the two $\mathrm{SL}(2,\mathbb{Z})$ transformation rules (1.3) and (1.9) relate to each other.

One resolution of the puzzle comes in the following way. First, one should notice that although the modulus ϕ appears naturally in the proof of $\mathrm{SO}(d,d,\mathbb{Z})$ Morita invariance of NCSYM, one can always exclude it from the final formula: its only contribution to the action functional (1.7) is through topological terms, and consequently the energy for $\phi \neq 0$ can be deduced from the one at $\phi = 0$ through

$$E(\phi) = E(\phi = 0) + \frac{V}{4g_{YM}^2}\Big(2\Phi_{ij}\mathrm{Tr}\ F^{ij} + (\Phi_{ij})^2 \mathrm{Tr}\ 1\Big) \tag{1.13}$$

We have $\mathrm{Tr}\ 1 = \dim E = p - q\theta$, and the central charge $\mathrm{Tr}\ F_{ij}$ can be calculated in terms of topological numbers by means of the formula relating the generating function of topological numbers $\mu(E)$ to the Chern character ch(E), which for $d = 2$ implies that $\mathrm{Tr}\ F_{ij} = q\epsilon_{ij}$. In this way, we recover the formula (1.8). Using the above formula, we can thus easily derive the transformation rule for the energy at $\phi = 0$:

$$E(\phi = 0) \to E(\phi = 0) + c\frac{2(cp + dq) - c|p - q\theta|}{2V g_{YM}^2 (c\theta + d)} \tag{1.14}$$

the rest term being strictly invariant.

Let us now consider the M-theory side. The exact mass formula $\mathcal{M}^2$ is invariant under the standard T-duality (1.3). In the limit $R_s \to 0$ and therefore $T_2 \to 0$, the transformation rules (1.3) reduce at leading order to the NCSYM ones (1.9). Under these truncated transformation rules, the rest term is thus invariant, but the quadratic term changes as in (1.14). However, there are subleading corrections to the transformation rules,

$$\begin{aligned} T_1 &\to \frac{aT_1+b}{cT_1+d} + \frac{cT_2^2}{(cT_1+d)^3} + \cdots \\ T_2 &\to \frac{T_2}{(cT_1+d)^2} - \frac{c^2T_2^3}{(cT_1+d)^4} + \cdots \end{aligned} \tag{1.15}$$

such that the rest term is no more invariant, but instead varies in the opposite way as the quadratic term (1.14), thus cancelling the anomalous variation of the energy. In this way, we have reconciled the two apparently different transformation rules (1.3) and (1.9).

M-theory and NCSYM on higher-dimensional tori. We now would like to show that this correspondence holds in more general cases. The mass formula for M-theory on a torus T^d times a circle is known in full generality from [12]. In order to keep things manageable and given the difficulties in defining the Matrix gauge theory on T^d, $d > 4$, we shall restrict ourselves to $d \leq 4$. In addition to the charges appearing in (1.1), we need to introduce those coming from the reduction of the charges m^{ijklm} and K^{ijkl} of the particle (or flux) and string (or momentum) multiplets on T^{d+1}. We should also take into account the vev $\mathcal{C}_3$ of the M-theory three-form on the torus T^d. The BPS mass formula then reads

$$\begin{aligned} \mathcal{M}^2 =& \frac{1}{R_s^2}(\tilde{m}_s)^2 + (\tilde{m}_i)^2 + \frac{(m^{ij})^2}{2l_M^6} + \frac{R_s^2}{l_M^6}(\tilde{m}^{is})^2 + \frac{R_s^2}{4!\ l_M^{12}}(\tilde{m}^{ijkls})^2 + \\ &+ \sqrt{\frac{(\tilde{K}^i)^2}{l_M^6} + \frac{R_s^2(\tilde{K}^s)^2}{l_M^6} + \frac{(\tilde{K}^{ijkl})^2}{4!\ l_M^{12}} + \frac{R_s^2(\tilde{K}^{ijks})^2}{3!\ l_M^{12}}} \end{aligned} \tag{1.16}$$

where the shifted charges include the effect of the moduli through

$$\begin{aligned} \tilde{m}_s &= m_s + \frac{1}{2}B_{ij}m^{ij} + \frac{1}{8}B_{ij}B_{kl}m^{ijkls}\,, & \tilde{m}_i &= m_i + A_i m^s + B_{ij}m^{js} + \frac{1}{2}\mathcal{C}_{ijk}m^{jk} \\ \tilde{m}^{is} &= m^{is} + A_i m^{ij} + \frac{1}{3!}\mathcal{C}_{jkl}m^{ijkls}\,, & \tilde{m}^{ij} &= m^{ij} + \frac{1}{2}B_{kl}m^{ijkls} \\ \tilde{m}^{ijks} &= m^{ijks} + A_l m^{ijkls}\,, & \tilde{m}^{ijkls} &= m^{ijkls} \\ \tilde{K}^s &= K^s + A_i K^i + \frac{1}{3!}\mathcal{C}_{ijk}K^{ijks}\,, & \tilde{K}^i &= K^i + \frac{1}{2}B_{jk}K^{ijks} + \frac{1}{3!}\mathcal{C}_{jkl}K^{ijkl} \\ \tilde{K}^{ijks} &= K^{ijks} + A_l K^{ijkl}\,, & \tilde{K}^{ijkl} &= K^{ijkl} \end{aligned} \tag{1.17}$$

The composite charges K of the string multiplet are as usual expressed for 1/4-BPS states in terms of those in the particle multiplet through

$$K^s = m_i m^{is}\,, \quad K^i = m_j m^{ij} + m_s m^{is}\,, \quad K^{ijks} = m_l m^{ijkls}\,, \quad K^{ijkl} = m_s m^{ijkls} \tag{1.18}$$

and vanish for half-BPS states. In the Sen-Seiberg limit, the only terms remaining are $(\tilde{m}^s)^2$ at leading order, $(\tilde{m}^{ij})^2$ and $(\tilde{m}^{is})^2$ at next-to-leading order, together with the first and third term under the square root. We thus obtain, in string variables,

$$\begin{aligned}\mathcal{M} =& \frac{|\tilde{m}_s|}{g_s l_s} + \frac{g_s l_s}{2|\tilde{m}_s|}(\tilde{m}_i)^2 + \frac{1}{2\cdot 2 g_s l_s^5 |\tilde{m}_s|}(\tilde{m}^{ij})^2 + \frac{g_s}{2 g_s l_s^3 |\tilde{m}_s|}(\tilde{m}^{is})^2 + \\ &+ \frac{1}{2 l_s^3 |\tilde{m}_s|}\sqrt{(\tilde{K}^i)^2 + \frac{1}{4!\; g_s^2 l_s^6}(\tilde{K}^{ijkl})^2}\end{aligned} \tag{1.19}$$

On the other hand, the energy spectrum for NCSYM on a torus T^d was derived in full rigour in [14] for up to $d=4$ by computing the central charges in the superalgebra. For $d=3$, it is possible to introduce a topological term $\lambda^{ijk}\mathrm{Tr}\ F_{0i}F_{jk}$ in the action which only affects charge quantization. One then finds

$$\begin{aligned}E =& \frac{g_{YM}^2}{2V\dim E}\left(n^i + \Theta^{ij}m_j + \lambda^i \dim E + \lambda^{ijk}q_{jk}\right)^2 + \\ &+ \frac{V}{4g_{YM}^2 \dim E}\left(q_{ij} + \dim E\ \Phi_{ij}\right)^2 + \frac{1}{\dim E}\sqrt{(k_i)^2}\end{aligned} \tag{1.20}$$

where $k_i = (m_i \dim E - q_{ij}(n^j + \Theta^{jk}m_k))$ and the contractions are performed with respect to the metric g_{ij} or its inverse g^{ij}. For $\Phi_{ij}=0$, this formula is in complete agreement with the M-theory side (1.19) under the standard map (1.12), upon identifying

$$\lambda^{ijk} = \mathcal{C}_{ijk}\,, \qquad p = m_s\,, \qquad n^i = m_i\,, \qquad q_{ij} = m^{ij}\,, \qquad m_i = m^{is}\,, \tag{1.21}$$

and dropping the charge K^{ijkl} which vanishes by antisymmetry.

Finally let us consider the case $d=4$. The dimension now involves the second Chern class r through $\dim E = p + \frac{1}{2}\Theta^{ij}q_{ij} + \frac{1}{8}\epsilon_{ijkl}\Theta^{ij}\Theta^{kl}r$ and the BPS spectrum takes the form

$$\begin{aligned}E =& \frac{g_{YM}^2}{2V\dim E}\left[n^i + \lambda^i \dim E + \lambda^{ijk}(q + *\Theta r)_{jk} + \Theta^{ij}m_j\right]^2 + \\ &+ \frac{V}{4g_{YM}^2\dim E}\left[(q + *\Theta r)_{ij} + \dim E\ \Phi_{ij}\right]^2 + \frac{1}{\dim E}\sqrt{(k_i)^2 + \frac{1}{g_{YM}^4}(k^2)}\ .\end{aligned} \tag{1.22}$$

Here $k_i = (m_i \dim E - (q + *\Theta r)_{ij}(n^j + \Theta^{jk}m_k) + (*\lambda_3)_i(ps - \epsilon^{jklm}q_{jk}q_{lm}/8))$, $k = pr - \epsilon^{ijkl}q_{ij}q_{kl}/8)^2$ and $(*\lambda_3)_i = \frac{1}{3!}\epsilon_{ijkl}\lambda^{jkl}$. Again, for $\Phi_{ij}=0$ this is in perfect agreement with the M-theory mass formula up to the rest term, upon identifying $r = \epsilon_{ijkl}m^{ijkls}/4!$.

2. Magnetic background versus non-commutativity

The magnetic background ϕ as a compensator field. We now would like to offer a second resolution of the puzzle, which yields a much deeper insight into the role of the modulus ϕ. Let us first focus on the first term in the NCSYM energy (1.8), together with the rest term. We can rewrite the two contributions in a more suggestive way as

$$E_{rest} + E_{flux} = \frac{1}{l_s}\sqrt{\frac{V}{g_{YM}^4 l_s^4}}\left(\frac{|p-q\theta|}{\sqrt{t}} + \frac{t^{3/2}\left[q-\phi(p-q\theta)\right]^2}{2(p-q\theta)}\right), \tag{2.1}$$

where we defined $t = l_s^2/V$. This expression is as we argued exactly invariant under the Morita transformation rules (1.9), or in terms of the present variables

$$\theta \to \frac{a\theta+b}{c\theta+d}, \qquad \phi \to (c\theta+d)^2\phi - c(c\theta+d)\,, \qquad t \to t/(c\theta+d)^2 \tag{2.2}$$

leaving l_s and V/g_{YM}^4 invariant. In fact, the expression (2.1) arises as the two leading terms in the small t expansion of the square root of an hypothetical "relativistic" generalisation of (2.1),

$$\tilde{\mathcal{M}}^2 = \frac{V}{g_{YM}^4 l_s^6}\frac{\left[m-n\theta\right]^2 + t^2\left[n-\phi(m-n\theta)\right]^2}{t}\,, \tag{2.3}$$

where each term is separately invariant under the NCSYM duality (1.9). We claim that (2.3) is (part of) the mass formula of Born-Infeld theory on a non-commutative torus, and will return to this claim shortly.

The $\mathrm{SL}(2,\mathbb{Z})$ invariance of (2.3) seems to be realized in a quite different way from that of the standard $\mathrm{SL}(2,\mathbb{Z})$ invariant mass formula

$$\mathcal{M}^2 = \frac{V}{g_{YM}^4 l_s^6}\frac{|m-nT|^2}{T_2}\,, \tag{2.4}$$

which appears as the first term of (1.6). In order to see the relation between the two, we note that these mass formulae can be recast in the form

$$\mathcal{M}^2 = \begin{pmatrix} m & n \end{pmatrix} \cdot e^t e \cdot \begin{pmatrix} m \\ n \end{pmatrix}, \tag{2.5}$$

where e is an element of $\mathrm{SL}(2,\mathbb{R})$ in both cases:

$$e = \frac{1}{\sqrt{T_2}}\begin{pmatrix} 1 & -T_1 \\ 0 & T_2 \end{pmatrix}, \qquad \tilde{e} = \frac{1}{\sqrt{t}}\begin{pmatrix} 1 & -\theta \\ t\phi & (1-\theta\phi)t \end{pmatrix}. \tag{2.6}$$

Now, in contrast to e, $\tilde{e}$ is not an upper triangular matrix, but instead a LU product

$$\tilde{e} = \frac{1}{\sqrt{t}} \begin{pmatrix} 1 & 0 \\ t\phi & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\theta \\ 0 & t \end{pmatrix} . \tag{2.7}$$

This can be brought into an upper triangular form through a SO(2) rotation from the left, so that $\tilde{e}$ can be rewritten as

$$\tilde{e} = \Omega(\alpha) \cdot \frac{1}{\sqrt{t(1+t^2\phi^2)}} \begin{pmatrix} 1+t^2\phi^2 & -(1+t^2\phi^2)\theta + t^2\phi \\ 0 & t \end{pmatrix} , \tag{2.8}$$

with

$$\Omega(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} , \qquad \sin\alpha = \frac{t\phi}{\sqrt{1+t^2\phi^2}}, \qquad \cos\alpha = \frac{1}{\sqrt{1+t^2\phi^2}} . \tag{2.9}$$

The rotation drops from the mass formula (2.5), so that the two mass formulae (2.3) and (2.4) agree upon identifying the two upper triangular $SL(2,\mathbb{R})$ elements e and $\Omega(\alpha)^{-1}\tilde{e}$, i.e.

$$T = \left(\theta - \frac{t^2\phi}{1+t^2\phi^2} \right) + i \frac{t}{1+t^2\phi^2} = \theta + it \cos\alpha e^{i\alpha} . \tag{2.10}$$

It is easy to check that under this change of variables, the Morita equivalence transformation rules (1.9) imply the usual T-duality ones (1.3). The interpretation of the modulus ϕ is thus quite clear: it, or rather its associated angle $\alpha = \arctan(t\phi)$, parametrizes the SO(2) maximal compact subgroup of $SL(2,\mathbb{R})$, which leaves the mass formula (2.5) invariant. This implies that the mass formula (2.3) is invariant under translations of α,

$$\alpha \to \alpha + \beta , \qquad t \to t \frac{\cos^2\alpha}{\cos^2(\alpha+\beta)} , \qquad \theta \to \theta + t \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha+\beta)} , \tag{2.11}$$

which identifies various θ, ϕ, t three-ples.[2] In particular, it is always possible to choose $\phi = 0$, upon redefining θ and t, or $\theta = 0$, upon redefining ϕ and t. At $\phi = 0$, the B-field is identified with the non-commutative deformation parameter θ, and the metric on the NCSYM side is simply the inverse of the M-theory metric, so that the gauge theory lives on the reciprocal torus, albeit non-commutative. At $\theta = 0$ instead, the map (2.10) yields $T = -1/(\phi + i/t)$, and recalling that $t = l_s^2/V$, we see that the gauge theory lives on the *commutative* torus which is T-dual to the M-theory torus. We should emphasize that the symmetry (2.11) holds only for the Born-Infeld mass formula (2.3), and not for its non-relativistic limit (2.1), and therefore only Born-Infeld theory should allow for this trade-off between non-commutativity and background magnetic field.

[2] The existence of such a symmetry was envisaged in [7], where it was rejected as unlikely.

Born-Infeld theory on a non-commutative two-torus. We can now include the effect of the other charges n^i, m^i and k_i into a relativistic generalisation of the NCSYM energy formula (1.8),

$$\tilde{\mathcal{M}}^2 = \left(\frac{V}{g_{YM}^2 l_s^4}\right)^2 (\tilde{p})^2 + (\tilde{n}^i)^2 + \left(\frac{1}{g_{YM}^2 l_s}\right)^2 (\tilde{q})^2 + \sqrt{\left(\frac{V}{g_{YM}^2 l_s^4}\right)^2 (k_i)^2}\,, \tag{2.12}$$

where the shifted charges incorporate the effect of the deformation parameter θ, the magnetic background ϕ and the topological angles λ^i through

$$\begin{aligned}
\tilde{p} &= p - \theta q \\
\tilde{q} &= q + \phi(p - \theta q) \\
\tilde{n}^i &= n^i - \theta m^i + \lambda^i(p - \theta q) \\
\tilde{m}^i &= m^i - \phi[n^i - \theta m^i + \lambda^i(q + \phi(p - \theta q))] + \lambda^i(p - \theta q)
\end{aligned} \tag{2.13}$$

This mass formula reduces to (1.8) in the Sen-Seiberg non-relativistic limit $l_s \to 0$ holding V/g_{YM}^2 and g_{ij}/l_s^2 fixed, and to the usual Born-Infeld energy formula when $\theta = 0$, and coincides with the one proposed in [26] in the special case $\phi = 0$. We claim that this is the BPS energy formula for supersymmetric Born-Infeld theory on a non-commutative torus. This theory may be tentatively defined by the action functional

$$S = \frac{V}{2g_{YM}^2}\mathrm{Tr}\ \sqrt{\det\left[g_{ij} + l_s^2(F_{ij} + \Phi_{ij}\cdot 1)\right]} + i\lambda^i \mathrm{Tr}\ F_{0i}\,, \tag{2.14}$$

but the determinant would need to be properly defined. The resulting theory should exhibit the same Morita equivalence like its well-defined NCSYM cousin. The invariance of the BPS mass formula under the SO(2) symmetry (2.11) hints to a dynamical symmetry of this putative theory, which would allow to trade an abelian magnetic background ($\phi \neq 0$) for a non-commutativity of the base-space ($\theta \neq 0$). Unfortunately, we are not able to prove this assertion at that stage. Given that non-abelian Yang-Mills on a commutative torus is Morita equivalent to NCSYM at a rational value of the deformation parameter θ, one may hope to use this symmetry to shed light on non-abelian Born-Infeld on a commutative torus by relating it to its abelian version on a commutative torus with magnetic background.

Generalization to higher-dimensional tori. We now would like to generalize our considerations to higher-dimensional tori. The standard T-duality invariant mass formula in the vector representation of $\mathrm{SO}(d,d,\mathbb{Z})$ takes the form

$$\mathcal{M}^2 = \begin{pmatrix} m_i & m^i \end{pmatrix} \cdot e^t e \cdot \begin{pmatrix} m_i \\ m^i \end{pmatrix}, \qquad e = \begin{pmatrix} (V^{-1})^t & \\ & V \end{pmatrix} \cdot \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} \cdot \tag{2.15}$$

where e is the general form of a $\mathrm{SO}(d,d,\mathbb{R})$ element up to left action of the maximal compact subgroup $\mathrm{SO}(d) \times \mathrm{SO}(d)$. V is the vielbein of the metric $G = V^t V$, and can be chosen in an upper triangular form. The action of $\mathrm{SO}(d,d,\mathbb{Z})$ then takes the standard form

$$G+B \to \big(\mathcal{A}(G+B)+\mathcal{B}\big)\big(\mathcal{C}(G+B)+\mathcal{D}\big)^{-1}, \qquad \begin{pmatrix} m_i \\ m^i \end{pmatrix} \to \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} m_i \\ m^i \end{pmatrix}, \tag{2.16}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are integer-valued matrices parametrizing an $\mathrm{SO}(d,d,\mathbb{Z})$ element,

$$\mathcal{A}^t\mathcal{C} = -\mathcal{C}^t\mathcal{A}\,, \qquad \mathcal{B}^t\mathcal{D} = -\mathcal{D}^t\mathcal{B}\,, \qquad \mathcal{A}^t\mathcal{D}+\mathcal{C}^t\mathcal{B}=1 \tag{2.17}$$

The experience with the $d=2$ case as well as the way Φ enters in the general energy formula (1.22) suggests that the appropriate element of $\mathrm{SO}(d,d)$ parametrized by the moduli g,Θ,Φ is instead

$$\tilde{e} = \begin{pmatrix} (v^{-1})^t & \\ & v \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \Phi & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \Theta \\ & 1 \end{pmatrix} \cdot \tag{2.18}$$

where v is the vielbein of the inverse metric $g^{-1}=v^t v$. This mass formula is invariant under the Morita equivalence transformation rule,

$$\begin{gathered} \Theta \to (\mathcal{A}\Theta+\mathcal{B})(\mathcal{C}\Theta+\mathcal{D})^{-1}, \qquad \Phi \to (\mathcal{C}\Theta+\mathcal{D})\Phi(\mathcal{C}\Theta+\mathcal{D})^t + \mathcal{C}(\mathcal{C}\Theta+\mathcal{D})^t, \\ g \to (\mathcal{C}\Theta+\mathcal{D})g(\mathcal{C}\Theta+\mathcal{D})^t, \qquad g_{YM}^2 \to \sqrt{|\det(\mathcal{C}\Theta+\mathcal{D})|}\,g_{YM}^2, \\ \begin{pmatrix} n^i \\ n_i \end{pmatrix} \to \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} n^i \\ n_i \end{pmatrix}. \end{gathered} \tag{2.19}$$

We now want to find a mapping between (g,Θ,Φ) and (G,B) which translates (2.19) into (2.16). For this, we rotate the element $\tilde{e}$ into a block upper triangular form through a $\mathrm{SO}(d)$ action from the left,

$$\tilde{e} = \begin{pmatrix} \frac{1+\Omega}{2} & \frac{1-\Omega}{2} \\ \frac{1-\Omega}{2} & \frac{1+\Omega}{2} \end{pmatrix} \cdot \begin{pmatrix} (1+v\Phi v^t)(v^{-1})^t & \\ & (1-v\Phi v^t)^{-1}v \end{pmatrix} \cdot \begin{pmatrix} 1 & \Theta-(g^{-1}-\Phi g\Phi)^{-1}\Phi g \\ & 1 \end{pmatrix}, \tag{2.20}$$

where the rotation parameter is given by the orthogonal matrix

$$\Omega = (1-v\Phi v^t)(1+v\Phi v^t)^{-1}. \tag{2.21}$$

The $\mathrm{SO}(d)$ rotation drops from the mass formula, so that we can identify (2.20) to (2.15),

$$G=(g-\Phi g^{-1}\Phi)^{-1}, \qquad B=\Theta-(g-\Phi g^{-1}\Phi)^{-1}\Phi g^{-1} \tag{2.22}$$

or, combining the symmetric and antisymmetric matrices,

$$G + B = \Theta + (g + \Phi)^{-1} \,. \tag{2.23}$$

This can be equivalently expressed in terms of the variable Ω,

$$G = v^t \frac{(\Omega+1)^2}{4\Omega} v \,, \qquad B = \Theta - v^t \frac{\Omega^2 - 1}{4\Omega} v \,. \tag{2.24}$$

Φ thus appears as a compensator field for the $\mathrm{SO}(d)$ compact subgroup of the continuous T-duality group $\mathrm{SO}(d,d,\mathbb{R})$.[3] Under this change of variables, the Morita equivalence (2.19) becomes the standard T-duality map (2.16). For $\Phi = 0$, the non-commutative deformation parameter Θ is equated to the M-theory B-field, while the metric is the inverse of the M-theory metric, so that the gauge theory lives in the deformed momentum space. For $\Theta = 0$ instead, the relation (2.24) becomes $G + B = (g + \Phi)^{-1}$, so that the gauge theory lives on the commutative T-dual torus. For $d = 2$, we recover the previous result (2.10) upon setting $v = \sqrt{t}$, $\Theta = i\theta$, $\Phi = i\phi$, $\Omega = e^{-i\alpha}$.

As in the $d = 2$ case, there is a continuous symmetry $\mathrm{SO}(d)$ which rotates $\tilde{e}$ and implies an action on the moduli v, Θ, Φ. The exponentiated action is cumbersome, but the infinitesimal transformation is easily derived,

$$dv = (\Omega+1)^{-2}\Omega d\Omega \; v \,, \qquad d\Theta = \frac{1}{4} v^t \left[\frac{\Omega - 1}{\Omega + 1} d\Omega + d\Omega^t \frac{\Omega - 1}{\Omega + 1} \right] v \,. \tag{2.25}$$

by requiring the left-hand side of the map (2.24) to stay invariant. Again, we stress that this symmetry is a property of the relativistic dispersion relation, and not of its galilean limit.

Born-Infeld theory on a non-commutative d-torus. In the same way as for $d = 2$, we can now use these results to derive relativistic generalizations of the BPS energy formulae (1.20), (1.22), which reduce to them in the decoupling limit $l_s \to 0$ and reproduce the $d = 2$ result (2.3) in the decompactification limit. One way to proceed is to start from the M-theory mass formula (1.16), translate it in gauge theory variables through the relations (2.24) at $\Phi = 0$, and switch on the modulus Φ according to the mass formula (2.18) translated into the appropriate representation of $\mathrm{SO}(d,d)$. Restricting ourselves to the case $d \leq 4$, we obtain

$$\begin{aligned} \tilde{\mathcal{M}}^2 = & \left(\frac{V}{g_{YM}^2 l_s^4} \right)^2 (\tilde{p})^2 + \frac{1}{l_s^4} (\tilde{m}^i)^2 + (\tilde{m}_i)^2 + \left(\frac{V}{g_{YM}^2 l_s^2} \right)^2 \frac{(\tilde{m}_{ij})^2}{2!} + \left(\frac{V}{g_{YM}^2} \right)^2 \frac{(\tilde{m}_{ijkl})^2}{4!} + \\ & + \sqrt{ \left(\frac{V}{g_{YM}^2 l_s^4} \right)^2 (\tilde{k}_i)^2 + \frac{1}{l_s^4} (\tilde{k})^2 + \left(\frac{V}{g_{YM}^2 l_s^2} \right)^2 \frac{(\tilde{k}_{ijk})^2}{3!} + \left(\frac{V l_s}{g_{YM}^2} \right)^4 \frac{(\tilde{k}_{ijkl})^2}{4!} } \end{aligned} \tag{2.26}$$

[3]The invariance under the other $\mathrm{SO}(d)$ compact subgroup is a consequence of the Lorentz invariance of the theory.

where the shifted charges incorporate the effect of the moduli:

$$\begin{cases} \tilde{p} = p + \frac{1}{2}\Theta^{ij}m^{ij} + \frac{1}{8}\Theta^{ij}\Theta^{kl}m_{ijkl} \\ \tilde{m}_{ij} = m_{ij} + \frac{1}{2}\Theta^{kl}m_{ijkl} + \frac{1}{2}\Phi_{ij}\tilde{p} \\ \tilde{m}_{ijkl} = m_{ijkl} - \frac{1}{2}\Phi_{[ij}\tilde{m}_{kl]} + \frac{1}{8}\Phi_{[ij}\Phi_{kl]}\,\tilde{p} \end{cases}$$

$$\begin{cases} \tilde{m}^i = m^i + \Theta^{ij}m_j + \lambda^{ijk}m_{jk} + \lambda^i\tilde{p} \\ \tilde{m}_i = m_i + \Phi_{ij}m^j + \frac{1}{3!}\lambda^{jkl}m_{ijkl} + \lambda^j\tilde{m}_{ij} \end{cases}$$

$$\begin{cases} \tilde{k}_i = k_i + \Theta^{jk}k_{ijk} + \frac{1}{3!}\lambda^{jkl}k_{ijkl} \\ \tilde{k}_{ijk} = k_{ijk} + \frac{1}{2}\Phi_{[ij}\tilde{k}_{k]} + \lambda^l\tilde{k}_{ijkl} \end{cases}$$

$$\tilde{k} = k + \frac{1}{3!}\lambda^{ijk}k_{ijk} + A^i\tilde{k}_i\,, \qquad \tilde{k}_{ijkl} = k_{ijkl} \tag{2.27}$$

Here we gathered the charges according to their representation under SO(4,4): conjugate spinor (p, m_{ij}, m_{ijkl}), vector (m^i, m_i), spinor (k_i, k_{ijk}), singlets (k, k_{ijkl}) respectively. For 1/4-BPS states, the momentum multiplet charges are related to the flux multiplet charges through

$$k = m_i m^i\,, \qquad k_i = m^j m_{ij} + m\; m_i\,, \qquad k_{ijk} = m^l m_{ijkl}\,, \qquad k_{ijkl} = m\; m_{ijkl} \tag{2.28}$$

and vanish for half-BPS states. The energy formula (2.26) is invariant under the Morita equivalence transformation (2.19) supplemented by the laws

$$\lambda^{ijk} \to \sqrt{|\det(\mathcal{K})|}\mathcal{K}^i_l\mathcal{K}^j_m\mathcal{K}^k_n\lambda^{lmn}\,, \tag{2.29a}$$

$$\lambda^i \to \sqrt{|\det(\mathcal{K})|}\mathcal{K}^i_j\lambda^j - [\mathcal{C}\mathcal{K}^{-1}]^j_k\sqrt{|\det(\mathcal{K})|}\mathcal{K}^i_l\mathcal{K}^j_m\mathcal{K}^k_n\lambda^{lmn} \tag{2.29b}$$

where $\mathcal{K} = ((\mathcal{C}\Theta + \mathcal{D})^t)^{-1}$, as appropriate for a spinor representation of SO(d,d), [15, 19, 14]. Besides, in contrast to its non-relativistic limit, it is by construction invariant under the full U-duality group, SO$(5,5,\mathbb{Z})$ in the maximal case $d = 4$ that we considered, which in the context of M(atrix) theory was named extended U-duality [21, 22, 23]. It is also invariant under the SO(d) symmetry (2.25) which allows to trade the deformation parameter Θ for the magnetic background Φ. For $\Phi = 0$, it agrees with the formula constructed in [26]. Equation (2.26) should therefore give the energy of 1/4-BPS states of supersymmetric Born-Infeld on a non-commutative torus T^d with arbitrary magnetic background Φ, once this theory receives an appropriate definition.

Note added in proof. After our work was released, a related work appeared on the archive [27], discussing the relevance of non-commutative gauge theories in the context of D-branes. In particular, the equivalence of Born-Infeld gauge theories on a non-commutative torus under continuous shifts of Θ was proved and the field redefinition explicitly constructed. This is closely related to the SO(d) symmetry uncovered in the present work on the base of consideration of the BPS spectrum. The relation (2.24) between the gauge theory and space-time moduli agrees with eq. (4.3) in [27], upon identifying our gauge theory metric g with their open string metric G, and our target-space metric and B-field $G + B$ with the T-dual of their closed string metric and B-field $g + B$, as it should since our discussion was from the point of view of D0-branes instead of Dd-branes.

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References

[1] A. Connes, M.R. Douglas, and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] M. R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[3] M. Li, *Comments on supersymmetric Yang–Mills theory on a noncommutative torus*, hep-th/9802052.

[4] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219]; *Constrained quantization of open string in background B field and noncommutative D-brane*, hep-th/9906192.

[5] F. Ardalan, H. Arfaei, and M.M. Sheikh-Jabbari, *Dirac quantization of open strings and noncommutativity in branes*, hep-th/9906161.

[6] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[7] C. Hofman and E. Verlinde, *U-duality of Born-Infeld on the noncommutative two torus*, *J. High Energy Phys.* **12** (1998) 010 [hep-th/9810116].

[8] A. Sen, *D0-branes on T^n and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [hep-th/9709220].

[9] N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[10] T. Banks, W. Fischler, S.H. Shenker, and L. Susskind, *M theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[11] L. Susskind, *Another conjecture about M(atrix) theory*, hep-th/9704080.

[12] N.A. Obers and B. Pioline, *U-duality and M theory*, hep-th/9809039. to appear in *Phys. Rept.*

[13] N.A. Obers and B. Pioline, *U-duality and M theory, an algebraic approach*, in *Proceedings of 2nd conference on quantum aspects of gauge theories, supersymmetry and unification*, Corfu, Greece, Sep. 1998. hep-th/9812139.

[14] A. Konechny and A. Schwarz, *Supersymmetry algebra and BPS states of super Yang-Mills theories on noncommutative tori*, *Phys. Lett.* **B 453** (1999) 23 [hep-th/9901077].

[15] A. Schwarz, *Morita equivalence and duality*, *Nucl. Phys.* **B 534** (1998) 720 [hep-th/9805034].

[16] P.-M. Ho, *Twisted bundle on quantum torus and BPS states in matrix theory*, *Phys. Lett.* **B 434** (1998) 41 [hep-th/9803166].

[17] D. Brace, B. Morariu, and B. Zumino, *Dualities of the matrix model from T duality of the type II string*, *Nucl. Phys.* **B 545** (1999) 192 [hep-th/9810099].

[18] A. Konechny and A. Schwarz, *BPS states on noncommutative tori and duality*, *Nucl. Phys.* **B 550** (1999) 561 [hep-th/9811159].

[19] D. Brace, B. Morariu, and B. Zumino, *T-duality and Ramond-Ramond backgrounds in the matrix model*, *Nucl. Phys.* **B 549** (1999) 181 [hep-th/9811213].

[20] F. Hacquebord and H. Verlinde, *Duality symmetry of $N = 4$ Yang–Mills theory on T^3*, *Nucl. Phys.* **B 508** (1997) 609 [hep-th/9707179].

[21] M. Blau and M. O'Loughlin, *Aspects of U-duality in matrix theory*, *Nucl. Phys.* **B 525** (1998) 182 [hep-th/9712047].

[22] N.A. Obers, B. Pioline, and E. Rabinovici, *M theory and U duality on T^d with gauge backgrounds*, *Nucl. Phys.* **B 525** (1998) 163 [hep-th/9712084].

[23] C.M. Hull, *Matrix theory, U duality and toroidal compactifications of M theory*, *J. High Energy Phys.* **10** (1998) 011 [hep-th/9711179].

[24] P.K. Townsend, lectures at Cargèse NATO School on *Progress in string theory and M-theory*, France, May 1999.

[25] N. Seiberg and E. Witten, talk presented at *New ideas in particle physcis and cosmology*, University of Pennsylvania, May 1999.

[26] C. Hofman and E. Verlinde, *Gauge bundles and Born-Infeld on the noncommutative torus*, *Nucl. Phys.* **B 547** (1999) 157 [hep-th/9810219].

[27] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, hep-th/9908142.

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